Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Back Paper Examination
Algebra III

Time: 9.00 am to 1.00 pm   January 4, 2013   Instructor: Bhaskar Bagchi

Remark : The maximum you can score is 100.

1. For an algebraic number α, the minimal polynomial of α is the primitive integral polynomial of least degree having α as a root. We say that α is an algebraic integer if its minimal polynomial is monic.

   (a) Prove that the minimal polynomial of α is unique.

   (b) If α is the root of a monic integral polynomial then show that α is an algebraic integer.

   (C) If α is an algebraic integer, then state and prove a necessary and sufficient condition (in terms of its minimal polynomial) for α to be a unit in the ring of algebraic integers.

   [5+10+10=25]

2. (a) Let G be an abelian group. Prove that G admits the structure of a Q-module iff for every positive integer n and every $x \in G$, there is a unique $y \in G$ such that $x = ny$.

   (b) Let $d_1 \mid d_2 \mid \cdots \mid d_k$ and $e_1 \mid e_2 \mid \cdots \mid e_l$ be positive integers. If the groups $\bigoplus_{i=1}^{k} Z/d_i\ Z$ and $\bigoplus_{j=1}^{l} Z/e_j\ Z$ are isomorphic then show that $k = l$ and $d_i = e_i$ for all i.

   (c) Find an explicit ring isomorphism from $(Z/3\ Z) \oplus (Z/5\ Z)$ onto $Z/15\ Z$.

   [5+10+10=25]

3. (a) Let R be the ring of all continuous functions from $[0, 1]$ into real numbers, with pointwise operations. For $x \in [0, 1]$, let $M = \{f \in R : f(x) = 0\}$. Prove that each M_x is a maximal ideal of R, and conversely, every maximal ideal of R equals M_x for a unique $x \in [0, 1]$.

   (b) Prove that $\mathbb{C}[X_1, \cdots, X_n]$ is countable dimensional as a $\mathbb{C}$-vector space, but $\mathbb{C}(X_1, \cdots, X_n)$ is uncountable dimensional as a $\mathbb{C}$-vector space.

   [15+10=25]

4. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix with entries from a commutative ring. Suppose BA is the $n \times n$ identity matrix and AB is the $m \times m$ identity matrix. Show that $m = n$. [10]

5. (a) Prove that every unit in the ring $Z[\omega]$ has modulus 1. (Here ω is a complex cube root of unity).

(b) Give an example of an algebraic number $\alpha \in \mathbb{C}$ such that not every unit of $Z[\alpha]$ has modulus one.

[10+5=15]